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Multimode minimum uncertainty squeezed states

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Abstract. The concept of single mode 'squeezed state', a minimum uncertainty state with reduced fluctuations in either of the two canonically conjugate variables, is generalised to an arbitrary number of modes. It is shown that the n-mode squeezed states are a subset of the generalised coherent states of $\mathrm{Sp}(2n{:}\,\mathbb{R})$.

1. Introduction

Recently research in such seemingly disparate areas as gravitational radiation detection and communications theory has motivated consideration of novel states for the electromagnetic field known as squeezed states (Yuen and Shapiro 1978, 1980, Shapiro *et al* 1979, Caves 1981). These states are single mode minimum uncertainty states (MUS) for which the fluctuations in one quadrature phase of the field are smaller than would occur for a coherent state (Yuen 1976).

These states may also be considered as MUS of the one-dimensional harmonic oscillator for which the fluctuations in position are smaller than the fluctuations that would occur in the ground state.

It is our intention in this paper to generalise the concept of a single mode squeezed state to the many mode case.

The key to such a generalisation is to realise that single mode squeezed states are generated from the vacuum by an element of the unitary representation of $\mathrm{Sp}(2{:}\,\mathbb{R})$ given by exponentiating the operators of the corresponding Lie algebra. The appropriate generalisation is then to consider the squeezed states as a subset of generalised coherent states for $\mathrm{Sp}(2n{:}\,\mathbb{R})$.

For a specified set of canonical coordinates, not all unitary representations of $\mathrm{Sp}(2\mathrm{n}{:}\,\mathbb{R})$ will generate MUS from the vacuum. In § 2 we derive the condition an element of the representation must satisfy in order that the state produced be a MUS.

We show that the multi-mode squeezed states are the direct product of single mode squeezed states each of which is generated from the ground Fock state $|0\rangle$, by a unitary operator whose generator is an element of the Cartan subalgebra of $\mathrm{Sp}(2n{:}\,\mathbb{R})$.

2. Minimum uncertainty states

Given a set of canonical coordinates $\{\hat{q}_i, \hat{p}_i\}$ $(i=1, n)$ the commutation relations are

$$[\hat{q}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}. \tag{2.1}$$

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We then define the variances in the state $|\psi\rangle$ by

$$v(\hat{q}_i) \equiv \langle\psi|\hat{q}_i^2|\psi\rangle - \langle\psi|\hat{q}_i|\psi\rangle^2 \qquad v(\hat{p}_i) \equiv \langle\psi|\hat{p}_i^2|\psi\rangle - \langle\psi|\hat{p}_i|\psi\rangle^2.$$

We wish to find the states $|\psi\rangle$ which minimise the uncertainty product $v(\hat{q}_i)v(\hat{p}_i)$. The states $|\psi\rangle$ are then the MUS with respect to the set $\{\hat{q}_i, \hat{p}_i\}$.

The standard result (Biedenharn and Louck 1981) is that $|\psi\rangle$ will be a MUS when

$$\hat{Q}_i|\psi\rangle = \lambda_i\hat{P}_i|\psi\rangle \qquad \lambda_i \in \mathbb{C} \tag{2.2}$$

and

$$\langle\psi|\hat{P}_i\hat{Q}_i + \hat{Q}_i\hat{P}_i|\psi\rangle = 0 \tag{2.3}$$

where

$$\hat{Q}_i \equiv \hat{q}_i - \langle\psi|\hat{q}_i|\psi\rangle \tag{2.4}$$

$$\hat{P}_i \equiv \hat{p}_i - \langle\psi|\hat{p}_i|\psi\rangle. \tag{2.5}$$

When these conditions hold we find

$$v(\hat{q}_i) = \tfrac{1}{2}\mathrm{i}\hbar\lambda_i \tag{2.6}$$

$$v(\hat{p}_i) = \tfrac{1}{2}\mathrm{i}\hbar/\lambda_i \tag{2.7}$$

and

$$v(\hat{q}_i)v(\hat{p}_i) = \tfrac{1}{4}\hbar^2. \tag{2.8}$$

Since $\hat{q}_i$, $\hat{p}_i$ are self-adjoint operators in the Hilbert space $L^2(-\infty, \infty)$, $v(\hat{q}_i)$ and $v(\hat{p}_i)$ are positive definite. The second minimum uncertainty condition, equation (2.3), then requires that λ_i be pure imaginary.

The operators $\hat{p}_i$, $\hat{q}_i$ together with the identity I form the Lie algebra of the Heisenberg–Weyl group $N(n)$. It is convenient to define bose operators by a complex extension of this Lie algebra

$$a_i \equiv (1/\sqrt{2\hbar})(\mu_i\hat{q}_i + \mathrm{i}\hat{p}_i/\mu_i) \tag{2.9a}$$

$$a_i^\dagger \equiv (1/\sqrt{2\hbar})(\mu_i\hat{q}_i - \mathrm{i}\hat{p}_i/\mu_i). \tag{2.9b}$$

Then $[a_i, a_i^\dagger] = I$. The usual harmonic oscillator annihilation and creation operators have $\mu_i = \sqrt{\omega_i}$, where ω_i is the fundamental frequency of the ith mode. For later use we also define the quadrature phase operators by (Caves 1981).

$$\hat{X}_1^i \equiv \tfrac{1}{2}(a_i + a_i^\dagger) = (\mu_i/\sqrt{2\hbar})\hat{q}_i \tag{2.10a}$$

$$\hat{X}_2^i \equiv (1/2i)(a_i - a_i^\dagger) = (1/\sqrt{2\hbar})\hat{p}_i/\mu_i. \tag{2.10b}$$

The eigenstates of the harmonic oscillator with Hamiltonian

$$H \equiv \hbar\sum_i \mu_i^2(a_i^\dagger a_i + \tfrac{1}{2}) \tag{2.11}$$

span an irreducible unitary representation of $N(n)$. Let us indicate these states by $|\{n_i\}\rangle \equiv \Pi_{i=1}^n |n_i\rangle$ where n_i is the number of quanta in the ith mode. In particular we have the ground state $|0\rangle$ for which

$$a_i|0\rangle = 0 \tag{2.12}$$

for every a_i.

Using equation (2.9*a*) we see that $|0\rangle$ is in fact a MUS with $\lambda_i = -\mathrm{i}/\mu_i^2$. The variances of $\hat{p}_i$ and $\hat{q}_i$ for the state $|0\rangle$ are then

$$v(\hat{q}_i) = \hbar/2\mu_i^2 \tag{2.13}$$

$$v(\hat{p}_i) = \tfrac{1}{2}\hbar\mu_i^2. \tag{2.14}$$

Let $D(\alpha_i)$ indicate elements of the unitary representation of $N(n)$ obtained by exponentiating the elements of the Lie algebra $\{a_i, a_i^\dagger, I\}$. The coherent states $|\alpha\rangle$ are then defined by (Glauber 1963a, b).

$$|\{\alpha_i\}\rangle \equiv D(\alpha_i)|0\rangle \tag{2.15}$$

where

$$D(\alpha_i) = \exp\left(\sum_i \alpha_i a_i^\dagger - \alpha_i^* a_i\right).$$

The coherent states $|\{\alpha_i\}\rangle$ are also MUS. To see this we make use of an alternative interpretation of $D(\alpha_i)$, suggested by the following identity

$$\langle\{\alpha_i\}|\hat{A}|\{\alpha_i\}\rangle \equiv \langle 0|\bar{\hat{A}}|0\rangle \tag{2.16}$$

where

$$\bar{\hat{A}} \equiv D^\dagger(\alpha_i)\hat{A}D(\alpha_i). \tag{2.17}$$

We then define an isomorphism of the set of operators D to the set of linear transformations $\mathscr{D}$ defined by

$$D^\dagger(\alpha_i)\hat{q}_i D(\alpha_i) \equiv \hat{q}_i\mathscr{D}(\alpha_i) = \hat{q}_i + A_i \tag{2.18a}$$

$$D^\dagger(\alpha_i)\hat{p}_i D(\alpha_i) \equiv \hat{p}_i\mathscr{D}(\alpha_i) = \hat{p}_i + B_i \tag{2.18b}$$

where

$$\alpha_i = (1/\sqrt{2\hbar})(\mu_i A_i + \mathrm{i}B_i/\mu_i). \tag{2.19}$$

It is then clear that under the displacements (2.18*a*, *b*) the variances of both $\hat{q}_i$ and $\bar{\hat{q}}_i$ in the state $|0\rangle$ are equal and thus $|\{\alpha_i\}\rangle$ are also MUS with variances given by equations (2.13) and (2.14).

The concept of a coherent state has been generalised to an arbitrary Lie group G (Perelomov 1972). We now summarise the essential elements of this generalisation.

Let T be an irreducible representation of G acting in the Hilbert space $\mathscr{H}$. If $|\psi_0\rangle$ is some fixed vector in this space, we define a subgroup $H \subset G$ by the set of elements $\{h\}$ such that $T(h)|\psi_0\rangle = \mathrm{e}^{\mathrm{i}\alpha(h)}|\psi_0\rangle$. We refer to H as the stationary subgroup of G. The states $|\psi_h\rangle = T(h)|\psi_0\rangle$, $h \in H$ are clearly physically equivalent. Furthermore the states $|\psi_g\rangle$ for all g which belong to one left co-set G on H also differ from each other only by a phase factor.

The generalised coherent states $|\psi_g(x)\rangle$ are obtained by selecting a representative element $g(x)$ in the element x of G/H corresponding to g.

The definition of coherent states makes no reference to the concept of a MUS.

It is one of the purposes of this paper to find the subset of generalised coherent states of $\mathrm{Sp}(2n{:}\,\mathbb{R})$ which are MUS for a given set of canonical coordinates. To do this we make use of the isomorphism of unitary representations of $\mathrm{Sp}(2n{:}\,\mathbb{R})$ and the linear canonical transformations of the given set of canonical variables.

3. Sp(2*n*: ℝ) and minimum uncertainty

The generators of Sp($2n$: ℝ) are given by the following $n(2n+1)$ bilinear operators (Moshinsky 1973)

$$H_i \equiv \tfrac{1}{2}(a_i^\dagger a_i + a_i a_i^\dagger) \qquad i = 1, \ldots n \tag{3.1a}$$

$$a_i^\dagger a_j = C_{ij} \qquad i \neq j,\ i, j = 1, \ldots n \tag{3.1b}$$

$$a_i^\dagger a_j^\dagger,\ a_i a_j \qquad i \leqslant j = 1, \ldots n. \tag{3.1c}$$

The n^2 operators H_i and C_{ij} are the generators of the U(n) subgroup of Sp($2n$: ℝ).

The Fock states $|\{n_i\}\rangle$ form two irreducible representations of Sp($2n$: ℝ), one involving $\Sigma_i n_i =$ even and the other with $\Sigma_i n_i =$ odd. In the even Fock space the state with $n_i = 0$ for all i is the ground state $|0\rangle$ of the Hamiltonian (2.11). We take $|0\rangle$ as the fixed base vector for the definition of the generalised coherent states of Sp($2n$: ℝ). There are no minimum uncertainty states in the representation with $\Sigma_i n_i =$ odd. We will thus restrict the discussion to the even basis only. The stationary subgroup is then seen to be U(n). Thus $G/H =$ Sp($2n$: ℝ)/U(n) and the generalised coherent states are produced by acting on the state $|0\rangle$ with the unitary operators generated by the $n(n+1)$ operators of equation (3.1c). However only a subset of these states are MUS as we shall shortly show.

Let U be the unitary representation of Sp($2n$: ℝ). We then define the matrices S by

$$\hat{\boldsymbol{z}}' \equiv U^\dagger \hat{\boldsymbol{z}} U = \hat{\boldsymbol{z}} S \tag{3.2}$$

where

$$\hat{\boldsymbol{z}} = \sqrt{2\hbar}(\hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_2) \tag{3.3}$$

and

$$\hat{\boldsymbol{X}}_i = (\hat{X}_i^1, \ldots \hat{X}_i^n).$$

The matrices S are linear canonical transformations, i.e. $S\kappa S^{\mathrm{T}} = \kappa$ where

$$\kappa = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The matrices S form a matrix representation of Sp($2n$: ℝ).

If we write

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.4}$$

where A, B, C, D are real $n \times n$ matrices, then for S to be canonical we require (Moshinsky 1973)

$$CB^{\mathrm{T}} = DA^{\mathrm{T}} - I \tag{3.5a}$$

$$BA^{\mathrm{T}} = AB^{\mathrm{T}} \tag{3.5b}$$

$$B^{\mathrm{T}}D = D^{\mathrm{T}}B. \tag{3.5c}$$

If we further restrict S to be orthogonal then in addition to the above conditions we require

$$A = D \tag{3.6a}$$

$$B = -C \tag{3.6b}$$

$$(A^{\mathrm{T}} - \mathrm{i}B^{\mathrm{T}})(A + \mathrm{i}B) = I. \tag{3.6c}$$

This orthogonal subgroup is the group $\mathrm{U}(n)$.

We now consider the following question. What are the unitary operators U for which the state $|\psi\rangle = U|0\rangle$ is a MUS with respect to $\hat{z}$?

The first minimum uncertainty condition (equation (2.2)) may be written as (Note: $\langle\psi|\hat{q}_i|\psi\rangle = \langle\psi|\hat{p}_i|\psi\rangle = 0$)

$$(\hat{\boldsymbol{X}}_1 + \mathrm{i}\hat{\boldsymbol{X}}_2\Lambda)|\psi\rangle = 0 \tag{3.7}$$

where Λ is the diagonal matrix $D(\lambda_1, \ldots \lambda_n)$. The second minimum uncertainty condition (equation (2.3)) requires that Λ be real.

Clearly if U is an element of the unitary representation of $U(n)$, $|\psi\rangle$ is a MUS with $\Lambda = I$, as it differs from the ground state only by a phase factor. Furthermore if U is an element of the unitary representation of $\mathrm{U}(n)$ the state $|\phi\rangle = U|\psi\rangle$, is a MUS with $\Lambda = I$, if and only if $|\psi\rangle$ is a MUS with $\Lambda = I$. This follows directly from the fact that $\mathrm{U}(n)$ is the stationary subgroup of $\mathrm{Sp}(2n\colon \mathbb{R})$.

Theorem 1. Let $|\psi\rangle = U|0\rangle$ be a generalised coherent state of $\mathrm{Sp}(2n\colon \mathbb{R})$. Then $|\psi\rangle$ is a MUS with respect to $\hat{z}$ if and only if a real diagonal matrix Λ exists such that

$$A = D\Lambda \qquad C = -B\Lambda \tag{3.8}$$

where A, B, C, D are real $n \times n$ matrices defined by equations (2.3) and (3.4).

Proof. Let $|\psi\rangle = U|0\rangle$ where U is an element of the unitary representation of $\mathrm{Sp}(2n\colon \mathbb{R})$. Assume $|\psi\rangle$ is MUS. Then there exists a real diagonal matrix Λ such that

$$(\hat{\boldsymbol{X}}_1 + \mathrm{i}\hat{\boldsymbol{X}}_2\Lambda)|\psi\rangle = 0$$

i.e.

$$(\hat{\boldsymbol{X}}_1 + \mathrm{i}\hat{\boldsymbol{X}}_2\Lambda)U|0\rangle = 0.$$

Acting from the left with $U^\dagger$ we find

$$U^\dagger(\hat{\boldsymbol{X}}_1 + \mathrm{i}\hat{\boldsymbol{X}}_2\Lambda)U|0\rangle = 0$$

$$\therefore\ (\hat{\boldsymbol{X}}'_1 + \mathrm{i}\hat{\boldsymbol{X}}'_2\Lambda)|0\rangle = 0$$

where $\hat{\boldsymbol{X}}'_1$ and $\hat{\boldsymbol{X}}'_2$ are defined by equation (3.2). Thus

$$(\hat{\boldsymbol{X}}_1(A + \mathrm{i}B\Lambda) + \mathrm{i}\boldsymbol{X}_2(D\Lambda - \mathrm{i}C))|0\rangle = 0$$

$$\therefore\ (\hat{\boldsymbol{X}}_1 + \mathrm{i}\hat{\boldsymbol{X}}_2(D\Lambda - \mathrm{i}C)(A + \mathrm{i}B\Lambda)^{-1})|0\rangle(A + \mathrm{i}B\Lambda) = 0$$

(we have placed the ket to the left of $A + \mathrm{i}B\Lambda$ as the matrices only act on the coordinates $\hat{\boldsymbol{X}}_i$). However as $|0\rangle$ is the ground state $(\hat{\boldsymbol{X}}_1 + \mathrm{i}\hat{\boldsymbol{X}}_2)|0\rangle = 0$, thus $(D\Lambda - \mathrm{i}C)(A + \mathrm{i}B\Lambda)^{-1} = I$ or $A = D\Lambda$ and $C = -B\Lambda$.

As a corollary to the above theorem we have the following. If $|\psi\rangle$ is a MUS for which $\Lambda = D(\lambda_1, \ldots \lambda_n)$ then the variances in the state $|\psi\rangle$ are

$$V(\hat{q}_i) = \hbar\lambda_i/2\mu_i^2 \tag{3.9a}$$

$$V(\hat{p}_i) = \hbar\mu_i^2/2\lambda_i. \tag{3.9b}$$

Proof. As $|\psi\rangle$ is a MUS $(\hat{\boldsymbol{X}}_1 + \mathrm{i}\hat{\boldsymbol{X}}_2\Lambda)|\psi\rangle = 0$ which can be written as $\hat{q}_i|\psi\rangle = -\mathrm{i}(\lambda_i/\mu_i^2)\hat{p}_i|\psi\rangle$. However as Λ is real $\langle\psi|\hat{q}_i\hat{p}_i + \hat{p}_i\hat{q}_i|\psi\rangle = 0$ which upon using the commutation relations (equation (2.1)) becomes $\langle\psi|\hat{q}_i\hat{p}_i|\psi\rangle = -\frac{1}{2}\mathrm{i}\hbar$. Thus $\langle\psi|\hat{q}_i^2|\psi\rangle = V(\hat{q}_i) = \frac{1}{2}\hbar\lambda_i/\mu_i^2$. Similarly $V(\hat{p}_i) = \hbar\mu_i^2/2\lambda_i$.

We define all those MUS for which $\Lambda \neq I$ as the multimode squeezed states. Thus the multimode squeezed states are a subset of the generalised coherent states of $\mathrm{Sp}(2n:\mathbb{R})$.

4. Two mode squeezed states

As an illustration of the preceding discussion we now discuss the two dimensional squeezed states in some detail.

Generalised coherent states of the semidirect product $N(2)\otimes\mathrm{Sp}(4:\mathbb{R})$ have been discussed by Gulshani and Volkov (1980), under the name of 'Heisenberg symplectic angular momentum coherent states'. Caves (1982) has also defined a set of two mode 'squeezed states'. The states considered by these authors however were not restricted to be MUS.

The ten generators of $\mathrm{Sp}(4:\mathbb{R})$ may be written in the form

$$\hat{T}_1 \equiv \tfrac{1}{2}(aa^\dagger + a^\dagger a) = (\hat{X}_1^2 + \hat{X}_2^2) \tag{4.1}$$

$$\hat{T}_2 \equiv \tfrac{1}{2}(bb^\dagger + b^\dagger b) = (\hat{Y}_1^2 + \hat{Y}_2^2) \tag{4.2}$$

$$\hat{T}_3 \equiv (a^\dagger b + b^\dagger a) = 2(\hat{X}_1\hat{Y}_1 + \hat{X}_2\hat{Y}_2) \tag{4.3}$$

$$\hat{T}_4 \equiv -\mathrm{i}(a^\dagger b - b^\dagger a) = 2(\hat{X}_1\hat{Y}_2 - \hat{X}_2\hat{Y}_1) \tag{4.4}$$

$$\hat{T}_5 \equiv (ab + a^\dagger b^\dagger) = 2(\hat{X}_1\hat{Y}_1 - \hat{X}_2\hat{Y}_2) \tag{4.5}$$

$$\hat{T}_6 \equiv -\mathrm{i}(ab - a^\dagger b^\dagger) = 2(\hat{X}_2\hat{Y}_1 + \hat{X}_1\hat{Y}_2) \tag{4.6}$$

$$\hat{T}_7 \equiv \tfrac{1}{2}(a^2 + a^{\dagger 2}) = (\hat{X}_1^2 - \hat{X}_2^2) \tag{4.7}$$

$$\hat{T}_8 \equiv -\tfrac{1}{2}\mathrm{i}(a^2 - a^{\dagger 2}) = (\hat{X}_1\hat{X}_2 + \hat{X}_2\hat{X}_1) \tag{4.8}$$

$$\hat{T}_9 \equiv \tfrac{1}{2}(b^2 + b^{\dagger 2}) = (\hat{Y}_1^2 - \hat{Y}_2^2) \tag{4.9}$$

$$\hat{T}_{10} \equiv -\tfrac{1}{2}\mathrm{i}(b^2 - b^{\dagger 2}) = (\hat{Y}_1\hat{Y}_2 + \hat{Y}_2\hat{Y}_1). \tag{4.10}$$

The generators $\hat{T}_1$ to $\hat{T}_4$ form a closed Lie algebra and are the generators of the stationary subgroup U(2). The generators $\hat{T}_8$, $\hat{T}_{10}$ comprise the Cartan sub-algebra of $\mathrm{Sp}(4:\mathbb{R})$.

The Unitary representation of $\mathrm{Sp}(4:\mathbb{R})$ is then defined by the exponential map

$$U_i(\gamma_i) \equiv \exp(-\mathrm{i}\gamma_i\hat{T}_i). \tag{4.11}$$

The representation space is once again taken to be the two-dimensional Fock space $|n_a, n_b\rangle$, with $n_a + n_b$ an even integer.

The symplectic matrices associated with U_i and $\hat{z} = (\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}})$ are defined by

$$\hat{z}' = U_i^\dagger \hat{z} U_i \equiv \hat{z} S_i(\gamma_i). \tag{4.12}$$

The explicit form of all the matrices $S_i(\gamma_i)$ is given in the appendix. It is clear that not all the transformations S_i are independent, in fact we have

$$S_7(\gamma_7) = S_1(\pi/4) S_8(-\gamma_7) S_1(-\pi/4) \tag{4.13}$$

$$S_9(\gamma_9) = S_2(\pi/4) S_{10}(-\gamma_9) S_2(-\pi/4) \tag{4.14}$$

$$S_5(\gamma_5) = S_3(\pi/4) S_8(-\gamma_5) S_{10}(-\gamma_5) S_3(-\pi/4) \tag{4.15}$$

$$S_6(\gamma_6) = S_3(\pi/4) S_7(\gamma_6) S_9(\gamma_6) S_3(-\pi/4) \tag{4.16}$$

$$S_4(\gamma_4) = S_3(-\pi/4) S_2(\gamma_4) S_1(-\gamma_4) S_3(\pi/4). \tag{4.17}$$

It is clear that the most general canonical transformation in two dimensions may be built out of the U(2) transformations together with the two scale changing transformations S_8 and S_{10} (equivalently U_8 and U_{10}).

Applying the conditions of equation (3.8) to the symplectic matrices it is clear that the two mode squeezed state in $\hat{z}$ is

$$|\gamma_8, \gamma_{10}\rangle \equiv U_8(\gamma_8) U_{10}(\gamma_{10}) |0\rangle. \tag{4.18}$$

For which

$$\Lambda = \begin{pmatrix} e^{2\gamma_8} & 0 \\ 0 & e^{2\gamma_{10}} \end{pmatrix} \tag{4.19}$$

and thus

$$V(\hat{X}_1) = \tfrac{1}{4} e^{2\gamma_8} \qquad V(\hat{X}_2) = \tfrac{1}{4} e^{-2\gamma_8} \tag{4.20a, b}$$

$$V(\hat{Y}_1) = \tfrac{1}{4} e^{2\gamma_{10}} \qquad V(\hat{Y}_2) = \tfrac{1}{4} e^{-2\gamma_{10}}. \tag{4.20c, d}$$

The two mode squeezed states are thus generated by the two operators of the Cartan sub-algebra of Sp(4: ℝ). The two mode squeezed states are thus the direct product of the single mode squeezed states for each mode.

Although the operators U_7, U_9, U_5, U_6 do not generate MUS from $|0\rangle$ with respect to $\hat{z}$, inspection of equations (4.13)–(4.16) suggest that the states generated will be MUS with respect to a different set of canonical variables $\hat{z}'$.

Determining for which variables a given state is a MUS is important in devising schemes to produce squeezed states of the electromagnetic field. Squeezed states may be generated in nonlinear optical processes such as parametric amplification (Walls 1983). In the non-degenerate case the interaction Hamiltonian for this process takes the form

$$H_I = \tfrac{1}{2} i\hbar \cdot \chi \cdot (ab - a^\dagger b^\dagger) \tag{4.21}$$

where χ is a coupling constant a, b are the annihilation operators for the two coupled modes. This interaction is clearly proportional to the generator $\hat{T}_6$. We thus expect the interaction to lead to a reduction of fluctuations in the variables $\hat{z}'$ where $\hat{z}' = \hat{z} S_3(\pi/4) S_1(\pi/4) S_2(\pi/4)$. In terms of bose variables of the new frame c, d this transformation may be written as

$$c\, e^{i\pi/4} = (a - ib)/\sqrt{2}$$

$$d\, e^{i\pi/4} = (b - ia)/\sqrt{2}.$$

Thus to observe the reduction of fluctuations one needs to combine the output modes a, b with a 90° phase shift between them.

5. Conclusion

We have shown that the n-mode squeezed states are a subset of the generalised coherent states of $Sp(2n\colon \mathbb{R})$. They are produced from the vacuum state $|0\rangle$ by unitary operators whose generators comprise the n-dimensional Cartan sub-algebra of $Sp(2n\colon \mathbb{R})$. Each element of this sub-algebra corresponds to a scale change in the canonical variables and generates a single mode squeezed state. Thus, for a given set of canonical variables, the multi mode squeezed states are the direct product of single mode squeezed states.

Using an isomorphism between the unitary representation of $Sp(2n\colon \mathbb{R})$ and a matrix representation with respect to a given set of canonical coordinates we have given a criteria (equations (3.8)) for determining the subset of generalised coherent states which are the squeezed states.

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Appendix

We give below the explicit form of the matrix representations of Sp(4).

$$S_1=\begin{pmatrix}\cos\gamma_1 & 0 & -\sin\gamma_1 & 0\\ 0 & 1 & 0 & 0\\ \sin\gamma_1 & 0 & \cos\gamma_1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\qquad S_2=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\gamma_2 & 0 & -\sin\gamma_2\\ 0 & 0 & 1 & 0\\ 0 & \sin\gamma_2 & 0 & \cos\gamma_2\end{pmatrix}$$

$$S_3=\begin{pmatrix}\cos\gamma_3 & 0 & 0 & -\sin\gamma_3\\ 0 & \cos\gamma_3 & -\sin\gamma_3 & 0\\ 0 & \sin\gamma_3 & \cos\gamma_3 & 0\\ \sin\gamma_3 & 0 & 0 & \cos\gamma_3\end{pmatrix}\qquad S_4=\begin{pmatrix}\cos\gamma_4 & \sin\gamma_4 & 0 & 0\\ -\sin\gamma_4 & \cos\gamma_4 & 0 & 0\\ 0 & 0 & \cos\gamma_4 & \sin\gamma_4\\ 0 & 0 & -\sin\gamma_4 & \cos\gamma_4\end{pmatrix}$$

$$S_5=\begin{pmatrix}\cosh\gamma_5 & 0 & 0 & -\sinh\gamma_5\\ 0 & \cosh\gamma_5 & -\sinh\gamma_5 & 0\\ 0 & -\sinh\gamma_5 & \cosh\gamma_5 & 0\\ -\sinh\gamma_5 & 0 & 0 & \cosh\gamma_5\end{pmatrix}$$

$$S_6=\begin{pmatrix}\cosh\gamma_6 & \sinh\gamma_6 & 0 & 0\\ \sinh\gamma_6 & \cosh\gamma_6 & 0 & 0\\ 0 & 0 & \cosh\gamma_6 & -\sinh\gamma_6\\ 0 & 0 & -\sinh\gamma_6 & \cosh\gamma_6\end{pmatrix}$$

$$S_7=\begin{pmatrix}\cosh\gamma_7 & 0 & -\sinh\gamma_7 & 0\\ 0 & 1 & 0 & 0\\ -\sinh\gamma_7 & 0 & \cosh\gamma_7 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\qquad S_8=\begin{pmatrix}e^{\gamma_8} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & e^{-\gamma_8} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$S_9=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cosh\gamma_9 & 0 & -\sinh\gamma_9\\ 0 & 0 & 1 & 0\\ 0 & -\sinh\gamma_9 & 0 & \cosh\gamma_9\end{pmatrix}\qquad S_{10}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & e^{\gamma^{10}} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & e^{-\gamma_{10}}\end{pmatrix}$$

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